NCERT SOLUTIONS
PHYSICS XI CLASS
CHAPTER - 7
SYSTEM OF PARTICLES AND ROTATIONAL MOTION

7.1 Give the location of the centre of mass of a (i) sphere, (ii) cylinder, (iii) ring, and (iv) cube, each of uniform mass density. Does the centre of mass of a body necessarily lie inside the body?

Sol. Geometric centre; No

The centre of mass (C.M.) is a point where the mass of a body is supposed to be concentrated. For the given geometric shapes having a uniform mass density, the C.M. lies at their respective geometric centres.

The centre of mass of a body need not necessarily lie within it. For example, the C.M. of bodies such as a ring, a hollow sphere, etc., lies outside the body.

7.2 In the HCl molecule, the separation between the nuclei of the two atoms is about 1.27 Å (1Å = 10^{-10} m). Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus.

Sol. We consider hydrogen atom at the origin.

$x_1 = 0$, $x_2 = 1.27$ Å and $m_2 = 35.5\, m_1$

m_2 is mass of Cl atom and m_1 is mass of H atom.

$$\therefore \quad X_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{0 \times m_1 + 35.5m_1 \times 1.27}{m_1 + 35.5m_1} = \frac{35.5 \times 1.27m_1}{36.6m_1} = 1.232 \text{ Å}$$

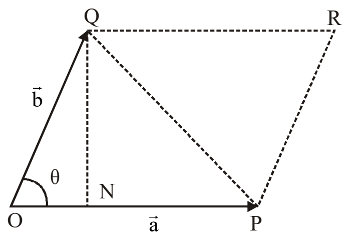
7.3 A child sits stationary at one end of a long trolley moving uniformly with a speed V on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, what is the speed of the CM of the (trolley + child) system?

Sol. No change

The child is running arbitrarily on a trolley moving with velocity v. However, the running of the child will produce no effect on the velocity of the centre of mass of the trolley.

This is because the force due to the boy's motion is purely internal. Internal forces produce no effect on the motion of the bodies on which they act. Since no external force is involved in the boy–trolley system, the boy's motion will produce no change in the velocity of the centre of mass of the trolley.

7.4 Show that the area of the triangle contained between the vectors $\vec{a}$ and $\vec{b}$ is one half of the magnitude of $\vec{a} \times \vec{b}$.



Sol. Let OP and OQ represents respectively.

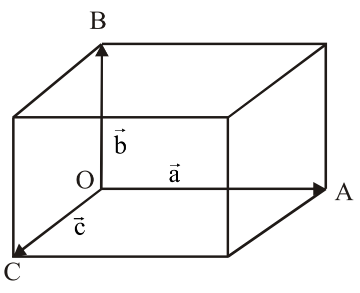
$\therefore \quad |\vec{a} \times \vec{b}| = OP \times OQ \sin\theta$

$|\vec{a} \times \vec{b}| = OP \times QN = 2$ (Area of Δ OPQ)

or Area of Δ OPQ = $\frac{1}{2}|\vec{a} \times \vec{b}|$

7.5 Show that $\vec{a}.(\vec{b} \times \vec{c})$ is equal in magnitude to the volume of the parallelepiped formed on the three vectors, $\vec{a}, \vec{b}$ & $\vec{c}$

Sol. Let three sides of a parallelopiped are represented



by three vectors $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$ and $\overrightarrow{OC} = \vec{c}$

Now, $\vec{b} \times \vec{c} = bc \sin\theta\ \hat{n} = bc \sin 90°\ \hat{n} = bc\ \hat{n}$

where $\hat{n}$ is a unit vector perpendicular to the plane

containing $\vec{b}$ & $\vec{c}$.

Now, $\vec{a}.(\vec{b} \times \vec{c}) = \vec{a}.bc\ \hat{n} = \vec{a}.\hat{n}\ bc = (a\ .\ 1 \cos 0)\ bc = abc$

7.6 Find the components along the x, y, z axes of the angular momentum L of a particle, whose position vector is $\vec{r}$ with components x, y, z and momentum is $\vec{p}$ with components p_x, p_y and p_z. Show that if the particle moves only in the x-y plane the angular momentum has only a z-component.

Sol. We know, $L = \vec{r} \times \vec{p} = (x\hat{i} + y\hat{j} + z\hat{k}) \times (P_x\hat{i} + P_y\hat{j} + P_z\hat{k})$

$$L_x\hat{i} + L_y\hat{j} + L_z\hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix}$$

or $L_x\hat{i} + L_y\hat{j} + L_z\hat{k} = \hat{i}\ (yP_z - zP_y) + \hat{j}\ (zP_x - xP_z) + \hat{k}\ (xP_y - yP_x)$

Comparing both sides

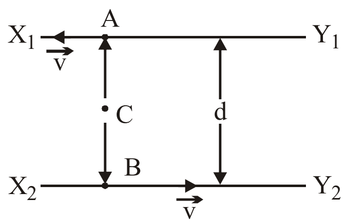
$L_x = yP_z - zP_y$

$L_y = zP_x - zP_z$

$L_z = xP_y - zP_x$

7.7 Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the vector angular momentum of the two particle system is the same whatever be the point about which the angular momentum is taken.

Sol. Angular momentum of two particle system about point A on X_1Y_1.



$\vec{L}_A = m\vec{v} \times 0 + m\vec{v} \times d = m\vec{v}d$

Similarly angular momentum of the two particle system about point B on X_2Y_2.

$\vec{L}_B = m\vec{v} \times 0 + m\vec{v}d$

Now, we consider a point C on AB such that AC = x

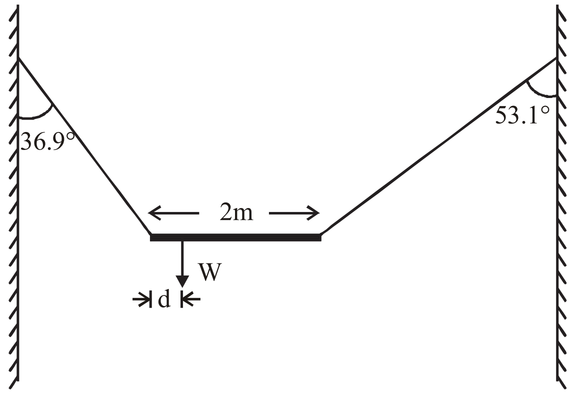
$\therefore$ BC = (d – x)

Angular momentum of the two particle system about C is

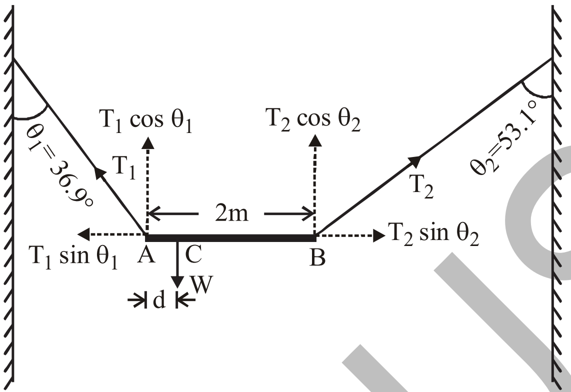
$\vec{L}_C = m\vec{v}\ (x) + m\vec{v}\ (d - x) = m\vec{v}d$

i.e., $\vec{L}_A = \vec{L}_B = \vec{L}_C$

7.8 A non-uniform bar of weight W is suspended at rest by two strings of negligible weight as shown in figure. The angles made by the strings with the vertical are 36.9° and 53.1° respectively. The bar is 2 m long. Calculate the distance d of the centre of gravity of the bar from its left end.



Sol. Let T_1 and T_2 are tensions in the two strings then for equilibrium in horizontal direction.



$T_1 \sin\theta_1 = T_2 \sin\theta_2$

or $\dfrac{T_1}{T_2} = \dfrac{\sin\theta_2}{\sin\theta_1} = \dfrac{\sin 53.1°}{\sin 36.9°} = \dfrac{0.7407}{0.5477} = 1.3523$

Let d be the distance of centre of gravity C of the bar from point A.

For equilibrium about point C

$T_1 \cos\theta_1 \,.\, d = T_2 \sin\theta_2 \,(2 - d)$

$T_1 \cos(36.9°)\, d = T_2 \cos 53.1° \,(2 - d)$

or $T_1 \times 0.8366\, d = T_2 \times 0.6718\,(2 - d)$

or $\dfrac{T_1}{T_2} \times 0.8366\, d = 0.6718\,(2 - d)$

or $\dfrac{0.7407}{0.5477} \times 0.8366\, d = 0.6718\,(2 - d)$ or $d = 0.72$ cm.

7.9 A car weighs 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.

Sol. The weight of the car acts at G. Let R_1 and R_2 be the forces exerted by the ground on the front wheel and back wheel respectively.

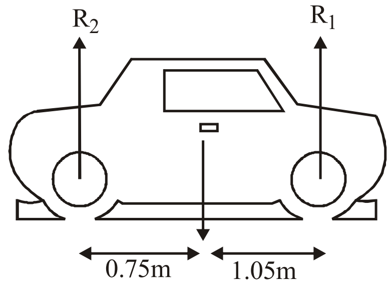
$\therefore\ R_1 + R_2 = W = Mg$

$R_1 + R_2 = 1800 \times 9.8$ N .... (1)

For equilibrium about point G

$R_1 \times 1.05 = R_2 \times 0.75$

or $\dfrac{R_1}{R_2} = \dfrac{0.75}{1.05} = \dfrac{5}{7}$ .... (2)



Using eq. (1) and (2), $\frac{5}{7}R_2 + R_2 = 1800 \times 9.8$

or $R_2 = \frac{7 \times 1800 \times 9.8}{12} = 102900$ N

$R_1 = \frac{5}{7}R_2 = \left(\frac{5}{7} \times 102900\right) = 7350$ N

$\therefore$ Reaction on each front wheel = $\frac{7305}{2} = 3675$ N

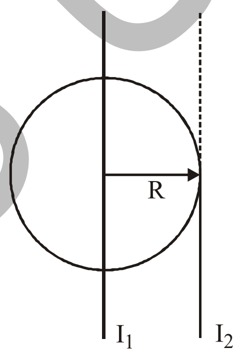
Reaction on each rear wheel = $\frac{10290}{2} = 5145$ N

7.10 (a) Find the moment of inertia of a sphere about a tangent to the sphere, given the moment of inertia of the sphere about any of its diameters to be $2MR^2/5$, where M is the mass of the sphere and R is the radius of the sphere.

(b) Given the moment of inertia of a disc of mass M and radius R about any of its diameters to be $MR^2/4$, find its moment of inertia about an axis normal to the disc and passing through a point on its edge.

Sol. (a) Applying parallel axes theorem

$$I_2 = I_1 + MR^2 = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$$



(b) Moment of inertia of the disc about any of its diameter

(i) Using theorem of perpendicular axes M.I. of the disc about an axis through its centre and normal to its plane.

$$I = I_D + I_D = \frac{MR^2}{4} + \frac{MR^2}{4} = \frac{MR^2}{2}$$

(ii) Using theorem of parallel axis. M.I. of the disc about ab axus passing through an edge point and normal to the disc

I' = M.I. of the disc about an axis passing through its centre

and perpendicular to its plane + $MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2$

7.11 Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time.

Sol. Let M and R be the mass and radius of the hollow cylinder and the solid sphere then

M.I. of hollow cylinder about its axis of symmetry

$$I_1 = MR^2$$

M.I. of sphere $I_2 = \frac{2}{5}MR^2$

Consider α_1 and α_2 be the angular acceleration in the cylinder and the sphere, when an external torque τ is applied then

$$\alpha_1 = \frac{\tau}{I_1} = \frac{\tau}{MR^2}, \ \alpha_2 = \frac{\tau}{I_2} = \frac{\tau}{\frac{2}{5}MR^2} = 2.5\frac{\tau}{MR^2} = 2.5\alpha_1 \quad \text{i.e. } \alpha_2 > \alpha_1$$

From first equation of rotational motion, $\omega = \omega_0 + \alpha t$

The angular speed of sphere will be more.

7.12 A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 rad s^{-1}. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?

Sol. Here, M = 20 kg, ω = 100 rad/s, R = 0.25 m

Moment of inertia of a solid cylinder is given by

$$I = \frac{1}{2}MR^2 = \frac{1}{2} \times 20 \times 0.25 \times 0.25 = 0.625 \text{ kg m}^2.$$

(a) Rotational K.E. = $I = \frac{1}{2}I\omega^2 = \frac{1}{2} \times 0.625 \times (100)^2 = 3125$ J

(b) Angular momentum L = Iω = 0.625 × 100 = 62.5 kgm^2/s

7.13 (a) A child stands at the centre of a turntable with his two arms outstretched. The turntable is set rotating with an angular speed of 40 rev/min. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value ? Assume that the turntable rotates without friction.

(b) Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do you account for this increase in kinetic energy?

Sol. (a) Here $\omega_1 = 40$ rpm, $I_2 = \frac{2}{5}I_1$

Using the principle of conservation of angular momentum

$$I_1\omega_1 = I_2\omega_2$$

or $\quad I_1 \times 40 = \frac{2}{5}I_1 \times \omega_2$ or $\omega_2 = 1000$ rpm

(b) Initial K.E. of rotation $= \frac{1}{2}I_1\omega_1^2$ or $E_i = \frac{1}{2}I_1(40)^2 = 800I_1$

Final K.E. of rotation $= \frac{1}{2}I_2\omega_2^2 = \frac{1}{2} \times \frac{2}{5}I_1(100)^2$

or $\quad E_f = 2000\,I_1 \quad \therefore \quad \frac{E_f}{E_i} = \frac{2000I_1}{800I_1} = 2.5$

7.14 A rope of negligible mass is wound round a hollow cylinder of mass 3 kg and radius 40 cm. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30 N? What is the linear acceleration of the rope? Assume that there is no slipping.

Sol. Here, M = 3 kg, R = 40 cm = 0.40 m, F = 30 N

Moment of inertia of the hollow cylinder is $I = MR^2 = 3 \times (0.4)^2 = 0.48$ kg m^2

We know, $\tau = F \times R = I\alpha$

$\therefore$ Angular acceleration, $\alpha = \frac{F \times R}{I} = \frac{30 \times 0.4}{0.48} = 25$ rad/s^2

Linear acceleration $a = R\,\alpha = 0.40 \times 25 = 10$ ms^{-2}

7.15 To maintain a rotor at a uniform angular speed or 200 rad/ s, an engine needs to transmit a torque of 180 N m. What is the power required by the engine ? Assume that the engine is 100% efficient.

Sol. Here, ω = 200 rad/s, τ = 180 Nm

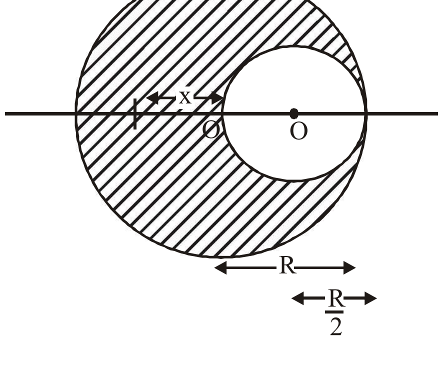
$\therefore$ Power = τ . ω = 180 × 200 = 36 × 10^3 W

7.16 From a uniform disk of radius R, a circular hole of radius R/2 is cut out. The centre of the hole is at R/2 from the centre of the original disc. Locate the centre of gravity of the resulting flat body.

Sol. Let mass per unit area of the disc be ρ.

Mass of disc = M = π R^2 ρ

Mass of scooped out part



$$m = \pi (R/2)^2 \rho = \frac{\pi R^2 \rho}{4} = \frac{M}{4}$$

If x in the distance of centre of mass form O then

$$x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{M\,(0) + m\,(R/2)}{M - m}$$

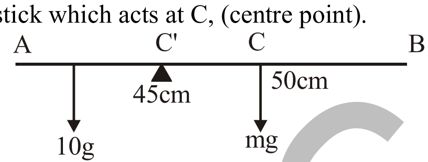
(M acts at 0 and m acts at 0')

$$= \frac{\frac{M}{4} \times \frac{R}{2}}{M - \frac{M}{4}} = \frac{-MR}{8} \times \frac{4}{3M} = \frac{-R}{6}$$

Position of centre of mass is (–R/6, 0).

7.17 A metre stick is balanced on a knife edge at its centre. When two coins, each of mass 5g are put one on top of the other at the 12.0 cm mark, the stick is found to be balanced at 45.0 cm. What is the mass of the metre stick?

Sol. Let m be the mass of the stick which acts at C, (centre point).



For equilibrium about C' i.e., 45 cm mark

$$10g\,(45 - 12) = mg\,(50 - 45)$$

$$10g \times 33 = mg \times 5$$

$$m = \frac{10 \times 33}{5} = 66\text{gm}$$

7.18 A solid sphere rolls down two different inclined planes of the same heights but different angles of inclination.

(a) Will it reach the bottom with the same speed in each case?

(b) Will it take longer to roll down one plane than the other?

(c) If so, which one and why?

Sol. Let v be the speed of the solid sphere at the bottom of the incline.

Applying principle of conservation of energy we get $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh$

$\frac{1}{2}mv^2$ is kinetic energy.

$\frac{1}{2}I\omega^2$ is rotational K energy.

But $I = \frac{2}{5}mr^2$ and $v = r\omega$

$$\therefore \quad \frac{1}{2}m\,(r^2\omega^2) + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\omega^2 = mgh \quad \text{or} \quad \frac{mr^2\omega^2}{2} + \frac{mr^2\omega^2}{5} = mgh$$

$$\frac{7}{10}v^2 = gh \quad \text{or} \quad v = \sqrt{\frac{10gh}{7}}$$

as h is same in both the cases v must be the same.

7.19 A hoop of radius 2m weighs 100 kg. It rolls along a horizontal floor so that its centre of mass has a speed of 20cm/s. How much work has to be done to stop it?

Sol. Here, R = 2m, M = 100 kg, v = 20 cm/s = 0.2 m/s

T.E. of the loop $= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$

$$= \frac{1}{2}Mv^2 + \frac{1}{2}(MR^2)\,\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{2}Mv^2 \qquad (v = r\omega)$$

$$= Mv^2 = 100\,(0.2)^2 = 4J$$

7.20 The oxygen molecule has a mass of 5.30×10^{-26} kg and a moment of inertia of 1.94×10^{-46} kg m^2 about an axis through its centre perpendicular to the lines joining the two atoms. Suppose the mean speed of such a molecule in a gas is 500 m/s and that its kinetic energy of rotation is two thirds of its kinetic energy of translation. Find the average angular velocity of the molecule.

Sol. Here, $m = 5.30 \times 10^{-26}$ kg, $I = 1.94 \times 10^{-46}$ kg m^2 , $v = 500$ m/s

Given, $\frac{1}{2}I\omega^2 = \frac{2}{3}\left(\frac{1}{2}mu^2\right)$

or $\frac{1}{2} \times (1.94 \times 10^{-46}) \times \omega^2 = \frac{1}{3} \times 5.30 \times 10^{-26} \times (500)^2$

or $\omega^2 = \frac{2 \times 5.3 \times 25}{3 \times 1.94} \times \frac{10^{-26} \times 10^4}{10^{-46}}$

or $\omega = 6.7 \times 10^{12}$ rad/s

7.21 A solid cylinder rolls up an inclined plane of angle of inclination 30°. At the bottom of the inclined plane the centre of mass of the cylinder has a speed of 5 m/s.

(a) How far will the cylinder go up the plane?

(b) How long will it take to return to the bottom?

Sol. Here, $a = \dfrac{g\sin\theta}{1 + \dfrac{K^2}{r^2}}$

For solid cylinder $K = r/\sqrt{2}$

$\therefore \quad a = \dfrac{g\sin 30°}{\dfrac{1 + (r\sqrt{2}\,)^2}{r^2}} = \dfrac{g \times 1/2}{1 + 1/2} = \dfrac{g}{3}$

From 3rd equation of motion

$v^2 = u^2 + 2as$

$0 = (5)^2 + 2\,(-g/3)\,L$

or $\dfrac{25 \times 3}{2g} = L$ or $L = 3.8$ ms

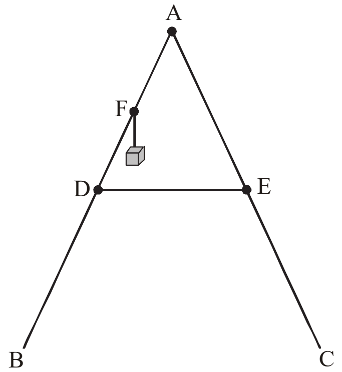
From 2nd equation of motion

$S = ut + \frac{1}{2}at^2$

$-L = 0 + \frac{1}{2}\left[\frac{-g}{3}\right]t^2$ or $t^2 = \frac{6L}{g} = \frac{6 \times 3.8}{9.8}$ or $t = 1.5$ sec.

$\therefore$ Total time for going up and coming down $= 2 \times 1.5 = 3.0$ sec.

7.22 As shown in figure, the two sides of a step ladder BA and CA are 1.6 m long and hinged at A. A rope DE, 0.5 m is tied half way up. A weight 40 kg is suspended from a point F, 1.2 m from B along the ladder BA. Assuming the floor to be frictionless and neglecting the weight of the ladder, find the tension in the rope and forces exerted by the floor on the ladder. (Take $g = 9.8$ m/s^2)



Sol. Considering the equilibrium of the system in the vertical direction, using S.I. unit

$R_1 + R_2 = W = 40 \times 9.8$

or $R_1 + R_2 = 40 \times 9.8$ .... (1)

Taking moments about A

$R_1 \times 0.5 = 40 \times 9.8 \times 0.125 + R_2 \times 0.5$

or $R_1 = 40 \times 9.8 \times 0.25 + R_2$

or $R_1 - R_2 = 40 \times 9.8 \times 0.25$ .... (2)

Adding eq. (1) and (2) gives

$2R_1 = 40 \times 9.8 \times 1.25$

$R_1 = 245$ N

Substituting for R_1 in eq. (1)

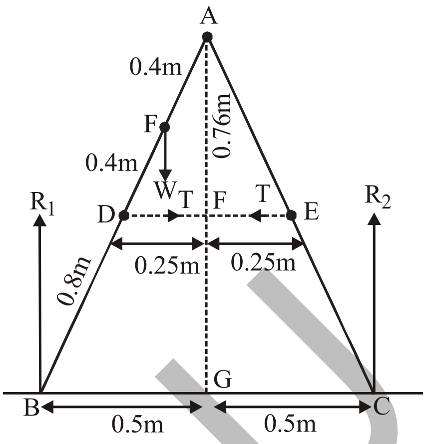
$R_2 = 40 \times 9.8 - 245 = 147$ N

Now considering the equilibrium of side CA along.

Taking moments about A

$T \times 0.76 = R_2 \times 0.5$

$$T = \frac{147 \times 0.5}{0.76} = 97 \text{ N}$$



7.23 A man stands on a rotating platform, with his arms stretched horizontally holding a 5 kg weight in each hand. The angular speed of the platform is 30 revolutions per minute. The man then brings his arms close to his body with the distance of each weight from the axis changing from 90cm to 20cm. The moment of inertia of the man together with the platform may be taken to be constant and equal to 7.6 kg m^2.

(a) What is his new angular speed? (Neglect friction.)

(b) Is kinetic energy conserved in the process? If not, from where does the change come about?

Sol. Here $I_1 = 7.6 + 2 \times 5\,(0.9)^2 = 15.7$ kg gm^2

$\omega_1 = 30$ rpm, $I_2 = 7.6 + (2 \times 5)\,(0.2)^2 = 8.0$ kg gm^2

According to principal of conservation of angular momentum, $I_1\omega_1 = I_2\omega_2$

$$\omega_2 = \frac{I_1\omega_1}{I_2} = \frac{15.7 \times 30}{8} \cong 59 \text{ rpm}$$

Kinetic energy in this process is not conserved.

As moment of inertia in this process decreases, K.E. of rotation increases, which comes out from the work done by the man in this process.

7.24 A bullet of mass 10 g and speed 500 m/s is fired into a door and gets embedded exactly at the centre of the door. The door is 1.0 m wide and weighs 12 kg. It is hinged at one end and rotates about a vertical axis practically without friction. Find the angular speed of the door just after the bullet embeds into it.

Sol. Angular momentum imparted by the bullet

$$L = mvr = (10 \times 10^{-3}) \times 500 \times \frac{1}{2} = 2.5 \text{ kg m}^2\text{/s}$$

Given $I = \dfrac{ML^2}{3} = \dfrac{12 \times (1.0)^2}{3} = 4$ kg m^2

From formula, $L = I\omega$

$$\omega = \frac{L}{I} = \frac{2.5}{4} = 0.625 \text{ rad/sec.}$$

7.25 Two discs of moments of inertia I_1 and I_2 about their respective axes (normal to the disc and passing through the centre), and rotating with angular speeds ω_1 and ω_2 are brought into contact face to face with their axes of rotation coincident. (a) What is the angular speed of the two-disc system? (b) Show that the kinetic energy of the combined system is less than the sum of the initial kinetic energies of the two discs. How do you account for this loss in energy? Take $\omega_1 \neq \omega_2$.

Sol. (a) Let I_1 and I_2 be the moments of inertia of two discs having angular speed ω_1 and ω_2.

When the two disc are in contact, the moment of inertia of the system will be $(I_1 + I_2)$.

Let its angular velocity is ω.

Applying law of conservation of angular momentum we have,

$I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\,\omega$

or $\quad \omega = \dfrac{(I_1\omega_1 + I_2\omega_2)}{(I_1 + I_2)}$

(b) Initial K.E. of the disc, $E_i = \dfrac{1}{2}I_1\omega_1^2 + \dfrac{1}{2}I_2\omega_2^2$

Final K.E., $\quad E_f = \dfrac{1}{2}(I_1 + I_2)\,\omega^2 = \dfrac{1}{2}(I_1 + I_2)\left[\dfrac{(I_1\omega_1 + I_2\omega_2)^2}{(I_1 + I_2)^2}\right]$

$\therefore$ Loss in energy $= E_i - E_f = \dfrac{1}{2}I_1\omega_1^2 + \dfrac{1}{2}I_2\omega_2^2 - \dfrac{(I_1\omega_1 + I_2\omega_2)^2}{2\,(I_1 + I_2)} = \dfrac{I_1 I_2\,(\omega_1 - \omega_2)^2}{2(I_1 + I_2)}$

The RHS of the above equation is positive i.e.,

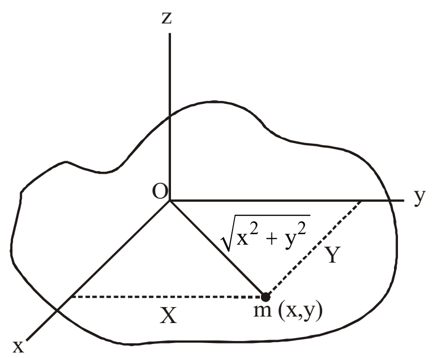
$E_i - E_f > 0$ or $E_i > E_f$

Thus K.E. of the combined system is less than the sum of the initial kinetic energy of the discs. This loss must be due to friction in the contact of the two discs.

7.26 (a) Prove the theorem of perpendicular axes.
(b) Prove the theorem of parallel axes.

Sol. (a) The theorem of perpendicular axes states that the moment of inertia of a planar body (lamina) about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.

A physical body with centre O and a point mass m,in the x–y plane at (x, y) is shown in the following figure.



Moment of inertia about x-axis, $I_x = mx^2$

Moment of inertia about y-axis, $I_y = my^2$

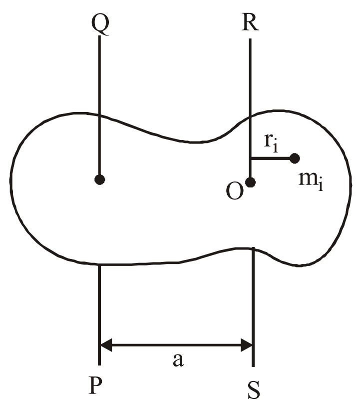
Moment of inertia about z-axis, $I_z = m(\sqrt{x^2 + y^2})^2$

$I_x + I_y = mx^2 + my^2 = m(x^2 + y^2) = m(\sqrt{x^2 + y^2})^2$

$I_x + I_y = I_z$

Hence, the theorem is proved

(b) The theorem of parallel axes states that the moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes. Suppose a rigid body is made up of n particles, having masses $m_1, m_2, m_3, \ldots, m_n$, at perpendicular distances $r_1, r_2, r_3, \ldots$, rn respectively from the centre of mass O of the rigid



body. The moment of inertia about axis RS passing through the point O:

$$I_{RS} = \sum_{i=1}^{n} m_i r_i^2$$

The perpendicular distance of mass m_i, from the axis $Q_P = a + r_i$

Hence, the moment of inertia about axis QP:

$$I_{QP} = \sum_{i=1}^{n} m_i \ (a + r_i)^2 = \sum_{i=1}^{n} m_i \ (a + r_i^2 + 2ar_i)^2$$

$$= \sum_{i=1}^{n} m_i a^2 + \sum_{i=1}^{n} m_i r_i^2 + \sum_{i=1}^{n} m_i \, 2ar_i = Ma^2 + I_{RS} + \sum_{i=1}^{n} m_i ar_i$$

[Σ m_i = M; M = Total mass of the rigid body]

Now, at the centre of mass, the moment of inertia of all the particles about the axis passing through the centre of mass is zero, that is,

$$2\sum_{i=1}^{n} m_i ar_i = 0 \qquad \because \ a \neq 0$$

$$\therefore \qquad \sum m_i r_i = 0$$

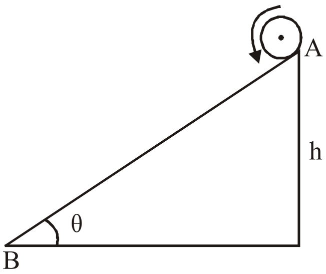
$$\therefore \qquad I_{QP} = I_{RS} + Ma^2$$

Hence, the theorem is proved

7.27 Prove the result that the velocity v of translation of a rolling body (like a ring, disc, cylinder or sphere) at the bottom of an inclined plane of a height h is given by $v^2 = \dfrac{2gh}{(1 + k^2 / R^2)}$.

Note k is the radius of gyration of the body about its symmetry axis, and R is the radius of the body. The body starts from rest at the top of the plane.

Sol. As the body is rolling down an incline plane we can apply law of conservation of energy.



Total energy at point B = Energy at point A

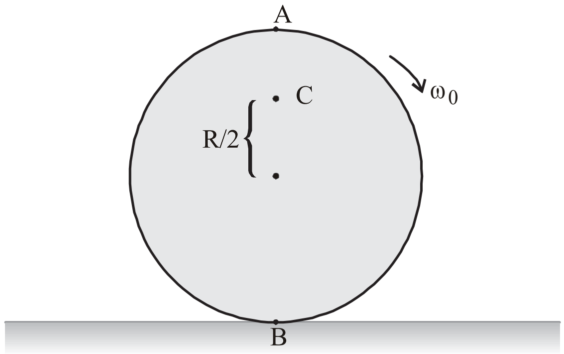
Translational K.E. + Rotational K.E. = Energy at point A

or $\quad \dfrac{1}{2}mv^2 + \dfrac{1}{2}I\omega^2 = mgh$

or $\quad \dfrac{1}{2}mv^2 + \dfrac{1}{2}(mk^2)\dfrac{v^2}{R^2} = mgh \qquad$ [as $\omega = v/R$]

or $\quad \dfrac{1}{2}mv\left(1 + \dfrac{k^2}{R^2}\right) = mgh \quad$ or $\quad v^2 = \dfrac{2gh}{(1 + k^2 / R^2)}$

7.28 A disc rotating about its axis with angular speed ω_0 is placed lightly (without any translational push) on a perfectly frictionless table. The radius of the disc is R. What are the linear velocities of the points A, B and C on the disc shown in figure? Will the disc roll in the direction indicated ?



Sol. We know, $v = R\omega$

For point, A $v_A = R\omega_0$ (same direction as arrow)

For point, B $v_B = R\omega_0$ (opposite direction as arrow)

For point, C $v_C = R\omega_0/2$ (same direction as arrow)

The disc is placed on a perfectly frictionless table so it will not rotate. Without friction, rolling is not possible.

7.29 Explain why friction is necessary to make the disc in Fig. (Q.7.28) roll in the direction indicated.

(a) Give the direction of frictional force at B, and the sense of frictional torque, before perfect rolling begins.

(b) What is the force of friction after perfect rolling begins?

Sol. A torque is required to roll the given disc. As per the definition of torque, the rotating force should be tangential to the disc. Since the frictional force at point B is along the tangential force at point A, a frictional force is required for making the disc roll.

(a) Force of friction acts opposite to the direction of velocity at point B. The direction of linear velocity at point B is tangentially leftward. Hence, frictional force will act tangentially rightward. The sense of frictional torque before the start of perfect rolling is perpendicular to the plane of the disc in the outward direction.

(b) Since frictional force acts opposite to the direction of velocity at point B, perfect rolling will begin when the velocity at that point becomes equal to zero. This will make the frictional force acting on the disc zero.

7.30 A solid disc and a ring, both of radius 10 cm are placed on a horizontal table simultaneously, with initial angular speed equal to 10 π rad s^{-1}. Which of the two will start to roll earlier ? The co-efficient of kinetic friction is $\mu_k = 0.2$.

Sol. Initial velocity of centre of mass is zero. Frictional force causes the CM to accelerate.

Friction force $f = \mu_k R = \mu_k mg$ when $R = mg$

$\therefore \quad \mu_k mg = ma \quad \text{or } a = \mu_k g$ .... (1)

From 1st equation of motion

$v = u + at$

$v = 0 + \mu_k gt$ .... (2)

The torque due to friction will decrease the initial angular speed ω_0 and hence produces angular retardation.

$\mu_k mg \times R = I\alpha \qquad (\tau = I\alpha)$

or $\quad \alpha = \dfrac{\mu_k mgR}{I}$ .... (3)

As $\omega = \omega_0 + \alpha\, t = \omega_0 - \dfrac{\mu_k Mg\, Rt}{I}$ .... (4)

Rolling begins when $v = R\omega$

Multiplying equation (4) by R and using eq. (2)

$$\mu_k mg = R\omega_0 - \frac{\mu_k mgR^2 t}{I}$$

For a Ring, $I = mR^2$

$\therefore \quad \mu_k gt = R\omega_0 - \mu_k gt$

or $\quad \omega_0 = \frac{2\mu_k gt}{R}$ or $t = \frac{\omega_0 R}{2\mu_k g}$

For a Disc, $I = \frac{1}{2}mR^2$

$\therefore \quad \mu_k gt = R\omega_0 - 2\mu_k gt$ or $t = \frac{\omega_0 R}{3\mu_k g}$

Now, $\omega_0 = 10\pi$ rad s^{-1}, $R = 0.1m$; $\mu_k = 0.2$

Ring will start rolling after a time, $t = \frac{\omega_0 R}{2\mu_k g} = \frac{10\pi \times 0.1}{2 \times 0.2 \times 9.8} = \frac{\pi}{9.8 \times 0.4} = 0.8s$

The disc will start rolling after a time

$$t_2 = \frac{\omega_0 R}{3\mu_k g} = \frac{10\pi \times 0.1}{3 \times 0.2 \times 9.8} = 0.53 \text{ s}$$

7.31 A cylinder of mass 10 kg and radius 15 cm is rolling perfectly on a plane of inclination 30°. The coefficient of static friction $\mu_s = 0.25$.

(a) How much is the force of friction acting on the cylinder?

(b) What is the work done against friction during rolling?

(c) If the inclination θ of the plane is increased, at what value of θ does the cylinder begin to skid, and not roll perfectly?

Sol. Here, m = 10 kg, r = 15cm = 0.15m, $\theta = 30°$, $\mu_s = 0.25$

(a) Force of friction is given by

$$f = \frac{mg\sin\theta}{3} = \frac{1}{3} \times 10 \times 9.8 \times \sin 30° = 16.3 \text{ N}$$

(b) The work done against friction during rolling is zero.

(c) For rolling without slipping, $\mu = \frac{\tan\theta}{3}$ or $\tan\theta = 3\mu = 3 \times 0.25 = 0.75$

or $\quad \theta = 37°$

7.32 Read each statement below carefully, and state, with reasons, if it is true or false;

(a) During rolling, the force of friction acts in the same direction as the direction of motion of the CM of the body.

(b) The instantaneous speed of the point of contact during rolling is zero.

(c) The instantaneous acceleration of the point of contact during rolling is zero.

(d) For perfect rolling motion, work done against friction is zero.

(e) A wheel moving down a perfectly frictionless inclined plane will undergo slipping (not rolling) motion.

Sol. (a) False

Frictional force acts opposite to the direction of motion of the centre of mass of a body.

In the case of rolling, the direction of motion of the centre of mass is backward. Hence, frictional force acts in the forward direction.

(b) True

Rolling can be considered as the rotation of a body about an axis passing through the point of contact of the body with the ground. Hence, its instantaneous speed is zero.

(c) False

When a body is rolling, its instantaneous acceleration is not equal to zero. It has some value.

(d) True

When perfect rolling begins, the frictional force acting at the lowermost point becomes zero. Hence, the work done against friction is also zero.

(e) True

The rolling of a body occurs when a frictional force acts between the body and the surface. This frictional force provides the torque necessary for rolling. In the absence of a frictional force, the body slips from the inclined plane under the effect of its own weight.

7.33 Separation of Motion of a system of particles into motion of the centre of mass and motion about the centre of mass:

(a) Show $p = p'_i + m_i V$

where p_i is the momentum of the ith particle (of mass m_i) and $p'_i = m_i v'_i$. Note v'_i is the velocity of the ith particle relative to the centre of mass.

Also, prove using the definition of the centre of mass $\Sigma p'_i = 0$.

(b) Show $K = K' + 1/2MV^2$.

where K is the total kinetic energy of the system of particles, K' is the total kinetic energy of the system when the particle velocities are taken with respect to the centre of mass and $MV^2/2$ is the kinetic energy of the translation of the system as a whole (i.e. of the centre of mass motion of the system).

(c) Show $L = L' + R \times MV$

where $L' = \Sigma r'_i + p'_i$ is the angular momentum of the system about the centre of mass with velocities taken relative to the centre of mass.

Remember $r'_i = r_i - R$; rest of the notation is the standard notation. Note L' and MR × V can be said to be angular momenta, respectively, about and of the centre of mass of the system of particles.

(d) Show $\frac{dL'}{dt} = \Sigma r_i' \times \frac{dp'}{dt}$

Further, show that $\frac{dL'}{dt} = \tau'_{ext}$

where τ'_{ext} is the sum of all external torques acting on the system about the centre of mass.

Sol. In this problems symbols used are

m_i = mass of i^{th} particle.

velocity of i^{th} particle

relative velocity of i^{th} particle w.r.t. CM.

velocity of centre of mass

K = K.E.

M = Total mass of the system

(a) (i) By definition relative velocity of i^{th} particle w.r.t. centre of mass is given by

$$\vec{V}_i' = \vec{V}_i - \vec{v} \Rightarrow \quad \vec{V}_i = \vec{V}_i' + \vec{v} \Rightarrow \quad m_i\vec{V}_i = m_i\vec{V}_i + m_i\vec{v}$$

$$\Rightarrow \quad \vec{P}_i = \vec{P}_i' + m_i\vec{v}$$

(ii) Now, $\Sigma\,\vec{P}_i' = \Sigma\, m_i\vec{V}_i' = \Sigma\, m_i\frac{d\vec{r}_i}{dt} = \frac{d}{dt}\Sigma\, m_i r_i = \frac{d}{dt}(0) = 0$

(b) We know, $K = K' + \frac{1}{2}Mv^2$

Now, $\vec{V}_i = \vec{V}_i' + \vec{v}_i$ or $\frac{1}{2}m_i\vec{v}_i^2 = \frac{1}{2}m_i(\vec{v}_i' + \vec{v}_0)^2$

or $\quad \Sigma\frac{1}{2}m_iV_i^2 = \Sigma\frac{1}{2}m_i(V_i'^2 + v_0^2 + 2\vec{v}_i'.\vec{v}) = \Sigma\frac{1}{2}m_iV_i'^2 + \Sigma\frac{1}{2}m_iv_0^2 + \Sigma m_i\vec{v}_i'.\vec{v}$

or $\quad K = K' + \frac{1}{2}Mv_0^2 + \vec{v}.\Sigma\, m_i\vec{V}_i = K' + \frac{1}{2}Mv_0^2 + \vec{v}.\Sigma\, P_i'$

But $\quad \Sigma P_i' = 0 \quad \therefore\ K = K' + \frac{1}{2}MV^2$

(c) We have to prove, $\vec{L} = \vec{L}' + \vec{R} \times M\vec{v}$

$\vec{L} =$ angular momentum of the system about any given point O (may be considered origin)

$\vec{L} = \Sigma\,(\vec{r}_i \times P_i')$ = angular momentum of the system about the CM with velocities taken relative to CM.

$\vec{R} =$ Position of C.M. w.r.t. O.

$\vec{v} =$ Velocity of CM

M = Total mass of the system of particles.

Also, $\vec{r}_i =$ Position vector of i^{th} particle w.r.t. O.

$\vec{r}_i' =$ Position vector of the particle w.r.t. CM.

Now, $\quad \vec{r}_i = \vec{r}_i' + \vec{R} \qquad$ .... (1)

$\vec{V}_i = V_i' + \vec{v} \qquad$ .... (2)

$\vec{P}_i = \vec{P}_i' + M\vec{v} \qquad$ .... (3)

So total angular momentum of the system about O is

$$\vec{L} = \Sigma\,(\vec{r}_i \times \vec{P}_i) = \Sigma\,[(\vec{r}_i' + \vec{R}) \times (\vec{P}_i' + m_i\vec{v})]$$

$$= \Sigma\,\vec{r}_i' \times P_i' + \vec{R} \times \Sigma\vec{P}_i' + \Sigma r_i' \times m_i v + \Sigma\vec{R} \times m_i\vec{v}$$

$$= \Sigma\,\vec{r}_i' \times P_i' + \vec{R} \times \Sigma\vec{P}_i' + \Sigma\,(r_i' + \vec{R}) \times m_i\vec{v} = L' + \vec{R} \times 0 + \Sigma\,(m_i\vec{r}_i) \times \vec{v}$$

When $\quad L' = \Sigma\vec{r}_i' \times \vec{P}_i'$ and $\Sigma P_i' = 0$

By definition of CM, $\vec{R} = \dfrac{\Sigma m_i\vec{r}_i}{M}$; $\Sigma m_i\vec{r}_i = \vec{R}M$

$\therefore \qquad L = L' + R \times M\vec{v}$

(d) (i) We have to prove, $\dfrac{dL'}{dt} = \Sigma\vec{r}_i' \times \dfrac{dP_i'}{dt}$

We have, $\quad L' = \Sigma\,(\vec{r}_i \times \vec{P}_i')$

$$\therefore \qquad \frac{dL'}{dt} = \Sigma\,\frac{d}{dt}[r_i' \times P_i'] = \Sigma\left[r_i' \times \frac{dP_i'}{dt} + \frac{dr_i'}{dt} \times P_i'\right]$$

Now, $\quad \vec{P}_i' = m_i\,\dfrac{dr_i'}{dt}$

$$\therefore \qquad \frac{dr_i'}{dt} \times P_i' = m_i\left[\frac{dr_i'}{dt} \times \frac{dr_i'}{dt}\right] = 0 \quad \therefore\ \frac{dL'}{dt} = \Sigma\,r_i' \times \frac{dP_i'}{dt}$$

(ii) $\dfrac{dL'}{dt} = \Sigma\,r_i' \times \dfrac{dP_i'}{dt}$

From Newton's second law

$\dfrac{dP_i'}{dt}$ = External force acting on i^{th} particle = $\vec{F}_i$

$\therefore \qquad \dfrac{dL'}{dt} = \Sigma\,r_i \times \vec{F}_i$ = Sum of all external torques acting on the system, about CM = $\vec{\tau}'_{ext}$.